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Feasible Controller Design for Stochastic Systems

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A control design method is presented for linear plant models with stochastic state disturbances and measurement noise such that time domain and frequency domain specifications are met. A time invariant linear quadratic Gaussian (LQG) controller structure is employed. The LQG controller is parametrized by the diagonal elements of the state weighting matrix Q and state noise intensity matrix L. A standard search algorithm is applied iteratively to compute Q and L such that all of the specifications are satisfied. At each iteration step of the search algorithm, an LQG problem is solved.

I. Introduction

IN this work the following problem is considered. A dynamic controller has to be designed for a linear plant model subject to stochastic state disturbances and measurement noise such that certain practical design specifications are satisfied.[1,2] Stochastic state disturbances are included to approximately model the random disturbances that exist in real systems. In addition, measurement noise is generated by sensors used to measure some of the system state variables. For safe and reliable system operation the controlled outputs and the control inputs should stay within certain limits. Engineers will thus want to design a controller that satisfies both time domain and frequency domain specifications and constraints. Within the framework of a linear stochastic system model, the time domain specifications will typically include constraints on the root mean square value of the controlled variables and the root mean variance of the control inputs. As far as frequency domain specifications are concerned, it would be realistic to consider constraints on the closed-loop poles as well as some bounds on the magnitude of given transfer functions. Various methods have been used in the past[3–5] to design controllers. However, it would be very difficult to design a controller satisfying the aforementioned specifications and constraints using these techniques.

For this reason a new method is presented that will facilitate the design of controllers with predictable performance. The main feature of the work presented here is that, instead of solving the aforementioned problem by some optimal control technique, the problem is solved by computing a feasible dynamic output feedback controller. The definition of feasible dynamic controllers and their computation for the problem posed earlier is roughly as follows: 1) The time invariant linear quadratic Gaussian (LQG) controller structure[6,7] is chosen as the dynamic controller. The LQG controller is then parametrized by a vector $p = [p_1, p_2, \ldots, p_{2m}]$ such that the state weighting matrix $Q = \mathrm{diag}(p_1^2, p_2^2, \ldots, p_m^2)$ and state noise intensity matrix $L = \mathrm{diag}(p_{m+1}^2, \ldots, p_{2m}^2)$, and where m is the order of the plant model. Note that the LQG controller formulation is used here as a design tool, and that no specific noise interpretation is given to the matrix L.[4] 2) A penalty function $J(p)$ that incorporates all the control specifications and all the constraints is constructed. The function $J(p)$ is contructed in such a manner that $J(p) \geq 0$ for all $p \in R^{2m}$, and it reaches the value of zero if and only if all the design specifications and all the constraints are all satisfied. A search algorithm[8,9] is then applied on the vector space in which p resides to bring the penalty function to zero. In other words, the equation $J(p) = 0$ is solved for the vector p. If a solution to this equation can be found, then the solution vector p^0 will parametrize a feasible LQG controller.

The paper is organized as follows. In Sec. II the plant model is presented, and in Sec. III the controller parametrization is given. The feasible control problem is formulated in Sec. IV, and the solution is given in Sec. V. The control design for a linearized vertical/short takeoff and landing (V/STOL) aircraft model is described in Sec. VI, and the relevant computational results are presented in Sec. VII.

II. Plant Model

Consider the state space description of a partially observable linear stochastic plant model[7,10] given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\,\mathrm{d}t + G\,\mathrm{d}\Phi(t) \tag{1}$$

$$\mathrm{d}y_\mu(t) = C_\mu x(t)\,\mathrm{d}t + H\,\mathrm{d}W(t) \tag{2}$$

$$y(t) = Cx(t) \tag{3}$$

where $x(t) \in R^m$ is the plant state, $u(t) \in R^q$ is the control input, $y_\mu(t) \in R^\nu$ is the measured output, $y(t) \in R^n$ is the controlled output, $n \leq q$, $\{\Phi(t),\ t \geq 0\}$ is an R^α-valued standard Wiener process representing the stochastic state disturbances, $\{W(t),\ t \geq 0\}$ is an R^β-valued standard Wiener process modeling the measurement noise, and the initial condition $x(0)$ is a random vector with $E(|x(0)|^2) < \infty$, where $E(\cdot)$ denotes the expectation operator and $|(\cdot)|$ stands for the magnitude of $(\cdot)$. It is assumed that $x(0)$, $\{\Phi(t),\ t \geq 0\}$, and $\{W(t),\ t \geq 0\}$ are stochastically independent. In addition, $A \in R^{m\times m}$, $B \in R^{m\times q}$, $G \in R^{m\times\alpha}$, $C \in R^{n\times m}$, $C_\mu \in R^{\nu\times m}$, and $H \in R^{\nu\times\beta}$.

III. Controller Model

In this work we assume that the plant model described by Eqs. (1) and (2) is controlled by a time invariant LQG controller[6,7,10] given by

$$\begin{aligned}\mathrm{d}x_h(t) &= (A - BF)x_h(t)\,\mathrm{d}t + \Pi[\mathrm{d}y_\mu(t) - C_\mu x_h(t)\,\mathrm{d}t] \\ &= [(A - BF - \Pi C_\mu)x_h(t) + \Pi C_\mu x(t)]\,\mathrm{d}t + \Pi H\,\mathrm{d}W(t)\end{aligned} \tag{4}$$

$$u(t) = -Fx_h(t) \tag{5}$$

$$F = R^{-1}B^T S \tag{6}$$

$$\Pi = \Xi C_\mu^T \Theta^{-1} \tag{7}$$

where $S = S^T \geq 0$ is the solution of the time invariant control algebraic Riccati equation

$$A^T S + SA + Q - SBR^{-1}B^T S = 0 \tag{8}$$

and where $\Xi = \Xi^T \geq 0$ is the solution of the time invariant filter algebraic Riccati equation

$$A\Xi + \Xi A^T + L - \Xi C_\mu^T \Theta^{-1} C_\mu \Xi = 0 \tag{9}$$

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In addition, $x_h(t) \in R^m$ is the compensator state, $R = R^T > 0$, $Q = Q^T \geq 0$, $L = L^T \geq 0$, $\Theta = \Theta^T > 0$, and we assume that (A, B) is controllable and (A, C_μ) is observable. The transfer function of the compensator is given by

$$\Lambda(s) = F(sI - A + BF + \Pi C_\mu)^{-1}\Pi \tag{10}$$

The poles of $\Lambda(s)$ are given by the eigenvalues of $(A - BF - \Pi C_\mu)$.

In the LQG problem, R is the control weighting matrix and Q is the state weighting matrix in the well-known integral quadratic cost function,[6] whereas $L = GG^T$ is the state noise intensity matrix and $\Theta = HH^T$ is the measurement noise intensity matrix [see Eqs. (1) and (2)]. However, in the problem considered here, the LQG controller formulation [see Eqs. (4–9)] is employed as a design tool and the matrices R, Q, L, and Θ are used as design parameters.[4,5] Thus, the LQG interpretation of R, Q, L, and Θ and the corresponding integral quadratic cost function are not directly applicable here. More specifically, no noise interpretation is given to the matrices L and Θ.

In this work the following structure is assumed for the parameter matrices:

$$R = I \tag{11}$$

$$\Theta = I \tag{12}$$

$$Q = \mathrm{diag}\left(p_1^2,\ p_2^2, \ldots,\ p_m^2\right) \tag{13}$$

$$L = \mathrm{diag}\left(p_{m+1}^2,\ p_{m+2}^2, \ldots,\ p_{2m}^2\right) \tag{14}$$

where $\{p_1, \ldots, p_{2m}\}$ are real numbers and $p = [p_1,\ p_2, \ldots,\ p_{2m}]$ becomes the controller parameter vector.

The advantage of choosing the matrices Q and L diagonal, $R = I$, and $\Theta = I$ is that considerable design flexibility is obtained while the number of design variables remains reasonable. Greater flexibility might be obtained by also using the off-diagonal elements of Q and L as design parameters. However, this will significantly increase the number of design variables. Roughly speaking, the choice of $R = I$ and $\Theta = I$ is also based on the following: 1) the form $R = \rho_0 I$, $\rho_0 > 0$, and $\Theta = \theta_0 I$, $\theta_0 > 0$, is sometimes used in the analysis and design of LQG controllers[11]; and 2) emphasis is placed here on the relative magnitude between R and Q and between Θ and L.

The state of the closed-loop system $z(t)$ consists of the state of the plant $x(t)$ and the state of the LQG controller $x_h(t)$,

$$z(t) = \left[x^T(t),\ x_h^T(t)\right]^T \in R^{2m} \tag{15}$$

In addition, define an $R^{\alpha+\beta}$-valued standard Wiener process,

$$\{P(t) = [\Phi^T(t),\ W^T(t)]^T,\ t \geq 0\} \tag{16}$$

Using Eqs. (1–5) and assuming a reference command input $r(t) \in R^n$, the closed-loop system turns out to be given by

$$\mathrm{d}z(t) = \left\{\begin{bmatrix} A & -BF \\ \Pi C_\mu & A - BF - \Pi C_\mu \end{bmatrix} z(t) + B_c r(t)\right\} \mathrm{d}t + \begin{bmatrix} G & 0 \\ 0 & \Pi H \end{bmatrix}\begin{bmatrix} \mathrm{d}\Phi(t) \\ \mathrm{d}W(t) \end{bmatrix}$$
$$= [A_c z(t) + B_c r(t)]\,\mathrm{d}t + M\,\mathrm{d}P(t) \tag{17}$$

$$y(t) = [C \quad 0]\, z(t) = C_c z(t) \tag{18}$$

and where $A_c \in R^{2m\times 2m}$, $M \in R^{2m\times(\alpha+\beta)}$, $C_c \in R^{n\times 2m}$, $B_c \in R^{2m\times n}$ and 0 denotes the zero matrix of appropriate dimensions in Eqs. (17) and (18). Here, we consider the case where the initial condition $z(0)$ is a random vector with given expected value $E(z(0)) = \bar{z}_0$ and variance $E((z(0) - \bar{z}_0)(z(0) - \bar{z}_0)^T) = \Sigma_0$. In addition, it can be shown that the closed-loop system poles are given by the eigenvalues of $(A - BF)$ and the eigenvalues of $(A - \Pi C_\mu)$ and are guaranteed stable.[6]

IV. Formulation of the Feasible Control Problem

We are interested in designing a dynamic compensator with a time invariant LQG controller structure as described in Sec. III such that the time domain and frequency domain specifications given later are satisfied. For computation purposes a time discretization $\Gamma = \{kT\}_{k=0}^{k=N}$ is used over a finite time horizon, where T is the sampling period.

Consider a series of different step reference command inputs each having the form $r^{(j)}(k) = c_j v^{(j)} \in R^n$, where $c_j \in R$ is a given constant, $v^{(j)} \in R^n$ is a unit vector along the jth axis in R^n, $j = 1, 2, \ldots, n$, and $k = 0, \ldots, N$. For each step reference command input $r^{(j)}(\cdot)$, denote the corresponding desired closed-loop response at discrete time instants $t_k \in \Gamma$, as $y_d^{(j)}(t_k) = y_d^{(j)}(k) = \psi_d^{(j)}(k) v^{(j)} \in R^n$, where $\psi_d^{(j)}(k) \in R$ is a given discrete time response, $j = 1, 2, \ldots, n$, and $k = 0, \ldots, N$. Then let the set Y be the collection of all step reference command input–desired closed-loop response pairs; that is, $Y = \{(r(\cdot), y_d(\cdot)) : r(\cdot) = r^{(j)}(\cdot), y_d(\cdot) = y_d^{(j)}(\cdot),\ j = 1, \ldots, n\}$. Consider a given reference command input–desired closed-loop response pair $(r(\cdot), y_d(\cdot)) \in Y$, and define the tracking error as

$$e(k) = y(k) - y_d(k), \qquad k = 0, \ldots, N \tag{19}$$

The time domain specifications in terms of the regulation of the tracking error are

$$|E(e_i(k))| \leq emax_i, \qquad i = 1, \ldots, n, \quad k = 0, \ldots, N \tag{20}$$

$$\left\{\frac{1}{N}\sum_{k=0}^{N} E\left(e_i^2(k)\right)\right\}^{\frac{1}{2}} \leq erms_i, \qquad i = 1, \ldots, n \tag{21}$$

where $emax_i > 0$ and $erms_i > 0$ are given numbers, $E(e_i^2(k)) = (E(e(k)e^T(k)))_{ii}$, $i = 1, \ldots, n$, and $k = 0, \ldots, N$. An additional time domain specification in terms of constraining the root mean variance of the control inputs is given by

$$\left(\frac{1}{N}\sum_{k=0}^{N} E\left(u_i^2(k)\right) - (E(u_i(k)))^2\right)^{\frac{1}{2}} \leq uvar_i, \qquad i = 1, \ldots, q \tag{22}$$

where $uvar_i > 0$, $i = 1, \ldots, q$, are given numbers, and $E(u_i^2(k)) - (E(u_i(k)))^2 = (E(u(k)u^T(k)) - E(u(k))E(u^T(k)))_{ii}$, $i = 1, \ldots, q$, and $k = 0, \ldots, N$.

In practical compensator design, some constraints have to be placed on the closed-loop poles. The constraints considered here are

$$|\lambda_i(A - BF)| \leq maxr, \qquad i = 1, \ldots, m \tag{23}$$

$$|\lambda_i(A - \Pi C_\mu)| \leq maxf, \qquad i = 1, \ldots, m \tag{24}$$

where $maxr > 0$ and $maxf > 0$ are given numbers and $\lambda_i(X)$ denotes the ith eigenvalue of a square matrix X. Further, robustness specifications[4,11,12] can be formulated in terms of a bound on the singular values of the return difference transfer function at the plant input

$$20\log_{10}(\sigma_i(I + T_{in}(j\omega))) \geq 20\log_{10}(s_{\min}), \qquad i = 1, \ldots, q,\ \omega > 0$$

where

$$T_{in}(j\omega) = \Lambda(j\omega)C_\mu(j\omega I - A)^{-1}B$$

and where $0 < s_{\min} < 1$ is a given number, $\sigma_i(X) = [\lambda_i\,(X^* X)]^{1/2}$ denotes the ith singular value of a matrix X such that $\sigma_1(X) \leq \sigma_2(X) \leq \cdots \leq \sigma_q(X)$ and $(\cdot)^*$ denotes the conjugate transpose of $(\cdot)$. In this work the preceding frequency domain constraint specified for all frequencies $\omega > 0$ is approximated by the same constraint specified at a finite number of discrete logarithmically spaced frequency points $\omega \in \Omega = \{\omega : \omega = 10^{k\Delta\zeta+\zeta_0},\ k = 0, 1, \ldots, N_\omega, \Delta\zeta > 0, \zeta_0 \in R\}$. The constraint can thus be written as

$$20\log_{10}(\sigma_i(I + T_{in}(j\omega))) \geq 20\log_{10}(s_{\min}), \quad i = 1, \ldots, q, \quad \omega \in \Omega \tag{25}$$

The problem dealt with here is as follows: find the controller parameter vector $p \in R^{2m}$ [that is, the matrices Q and L defined in terms of the elements of the vector p by Eqs. (13) and (14)] such that 1) the time domain constraints (20–22) are satisfied for all $(r(\cdot),\ y_d(\cdot)) \in Y$ and 2) the frequency domain constraints (23–25) are satisfied. The controller $\Lambda(s)$ obtained from the parameter vector p via Eqs. (6–14) will be called a feasible output feedback controller.

V. Solution of the Problem

For the solution of this problem we define the following penalty functions[8] for the corresponding constraints:

1) $G_1(x, a, b) = [\max(x - b, 0) + \max(a - x, 0)]^2$, $a < b$, for the constraint $a \leq x \leq b$, and where $\max(d_1, d_2)$ denotes the maximum between d_1 and d_2.

2) $G_2(x, a_1) = [\max(a_1 - x, 0)]^2$, for the constraint $x \geq a_1$.

3) $G_3(x, b_1) = [\max(x - b_1, 0)]^2$, for the constraint $x \leq b_1$.

If a given constraint is satisfied, then the value of the corresponding penalty function is zero. The functions G_1, G_2, and G_3 are used to penalize the square of the violations of design constraint (20), constraint (25), and constraints (21–24), respectively.

Define the following penalty function for the time domain constraints (20–22) for a given $(r(\cdot),\ y_d(\cdot)) \in Y$,

$$J_1(p,\ (r(\cdot),\ y_d(\cdot))) = \sum_{k=0}^{N}\sum_{i=1}^{n} a_1 G_1(E(e_i(k)),\ -emax_i,\ emax_i) + \sum_{i=1}^{q} a_2 G_3\left(\left(\frac{1}{N}\sum_{k=0}^{N} E\left(u_i^2(k)\right) - (E(u_i(k)))^2\right)^{\frac{1}{2}},\ uvar_i\right) + \sum_{i=1}^{n} a_3 G_3\left(\left\{\frac{1}{N}\sum_{k=0}^{N} E\left(e_i^2(k)\right)\right\}^{\frac{1}{2}},\ erms_i\right) \tag{26}$$

where $a_i > 0$, $i = 1, 2, 3$.

Also, define the following penalty function for the frequency domain constraints (23–25):

$$J_2(p) = \sum_{i=1}^{m} a_4 G_3(|\lambda_i(A - BF)|,\ maxr) + \sum_{i=1}^{m} a_5 G_3(|\lambda_i(A - \Pi C_m)|,\ maxf) + \sum_{\omega\in\Omega}\sum_{k=1}^{q} a_6 G_2(20\log_{10}(\sigma_k(I + T_{in}(j\omega))),\ 20\log_{10}(s_{\min})) \tag{27}$$

where $a_i > 0$, $i = 4, 5, 6$.

The overall penalty function incorporating all the control specifications and constraints is then

$$J(p) = \sum_{(r(\cdot), y_d(\cdot))\in Y} J_1(p,\ (r(\cdot),\ y_d(\cdot))) + J_2(p) \tag{28}$$

It can be seen that $J(p)$ is a function of the expected value of certain functions of the system's state and control input and a function of the eigenvalues of some matrices. The function $J(p)$ is constructed in such a manner that $J(p) \geq 0$ for all $p \in R^{2m}$, and it reaches the value of zero if and only if all the design specifications and all the constraints are all satisfied. A search algorithm[8,9] can then be applied on the vector space in which p resides to bring the penalty function to zero. In other words, the equation $J(p) = 0$ is solved for the vector p. If a solution to this equation can be found, then the solution vector p^0 will parametrize a feasible LQG controller. Thus, an important feature of the proposed method is that, whenever possible, only feasible controllers are computed for which all the system specifications and constraints are met.

In summary, the problem considered here is as follows: find a design parameter vector $p^0 \in R^{2m}$ such that $J(p^0) = 0$. Clearly, a vector $p^0 \in R^{2m}$ satisfies $J(p^0) = 0$ if and only if 1) the time domain constraints (20–22) are satisfied for all $(r(\cdot),\ y_d(\cdot)) \in Y$ and 2) the frequency domain constraints (23–25) are satisfied. Hence, the controller $\Lambda^0(s)$ obtained from p^0 via Eqs. (6–14) is a feasible output feedback controller.

The computation of p^0 was conducted by solving an unconstrained minimization problem on R^{2m}. This was done by using the well-known Nelder–Mead search algorithm as implemented by the function FMINS of the PC-MATLAB® Optimization Toolbox.[9] However, the mapping from p to $J(p)$ is too complicated to guarantee the convergence of the Nelder–Mead algorithm to a solution of $J(p) = 0$, $p \in R^{2m}$ (assuming that a solution to this equation exists).

The computation of $J(p)$ at each iteration of the search algorithm is done as follows. For a given design parameter p and a given pair $(r(\cdot),\ y_d(\cdot)) \in Y$, the matrices Q and L are computed using Eqs. (13) and (14). Then inserting in Eqs. (6–9) and solving for the gains F and Π, and subsequently solving Eqs. (39–46) (given later), the three terms on the right-hand side of Eq. (26) are computed yielding $J_1(p,\ (r(\cdot),\ y_d(\cdot)))$. All three terms on the right-hand side of Eq. (27) can be directly computed, yielding $J_2(p)$. For the time domain specifications the calculations are repeated for all $(r(\cdot),\ y_d(\cdot)) \in Y$ to obtain finally $J(p)$.

The computation of $E(e(k))$ and $E(e(k)e^T(k))$ requires the computation of $E(z(k))$ and $E(z(k)z^T(k))$, $k = 0, 1, \ldots, N$. To derive the necessary expressions, the closed-loop system given by Eqs. (17) and (18) must first be discretized as follows:

$$z(k+1) = A_{cd}z(k) + B_{cd}r(k) + N_d(k) \tag{29}$$

$$y(k) = C_{cd}z(k) \tag{30}$$

where

$$A_{cd} = e^{A_c T} \tag{31}$$

$$B_{cd} = \int_0^T e^{A_c\upsilon}\,\mathrm{d}\upsilon B_c \tag{32}$$

$$N_d(k) = \int_{kT}^{(k+1)T} \exp(A_c((k+1)T - \upsilon))M\,\mathrm{d}P(\upsilon) \tag{33}$$

$$C_{cd} = C_c \tag{34}$$

where the sequence $\{N_d(k),\ k = 0, 1, \ldots, N\}$ is a sequence of independent random vectors in R^{2m} with a Gaussian distribution and where

$$E(N_d(k)) = 0 \tag{35}$$

$$E\left(N_d(k)N_d^T(j)\right) = M_d\delta_{kj} = \left(\int_0^T e^{A_c\upsilon}MM^T e^{A_c^T\upsilon}\,\mathrm{d}\upsilon\right)\delta_{kj} \tag{36}$$

where $M_d \in R^{2m\times 2m}$ and $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise. Further, we have used $z(k)$ and $y(k)$ to denote $z(t_k)$ and $y(t_k)$, respectively, $k = 0, 1, \ldots, N$.

Now denote

$$m(k) = E(z(k)), \qquad k = 0, \ldots, N \tag{37}$$

$$V(k) = E(z(k)z^T(k)), \qquad k = 0, \ldots, N \tag{38}$$

Then, by using Eq. (29) it follows that

$$m(k+1) = E(z(k+1)) = E(A_{cd}z(k) + B_{cd}r(k) + N_d(k)) = A_{cd}m(k) + B_{cd}r(k), \qquad k = 0, \ldots, N-1 \tag{39}$$

Also, in a similar manner

$$V(k+1) = E(z(k+1)z^T(k+1))$$
$$= E\Big((A_{cd}z(k) + B_{cd}r(k) + N_d(k))$$
$$\times (A_{cd}z(k) + B_{cd}r(k) + N_d(k))^T\Big)$$
$$= A_{cd}V(k)A_{cd}^T + A_{cd}m(k)r^T(k)B_{cd}^T + B_{cd}r(k)m^T(k)A_{cd}^T$$
$$+ B_{cd}r(k)r^T(k)B_{cd}^T + M_d, \qquad k = 0, \dots, N-1 \tag{40}$$

The initial conditions for Eqs. (39) and (40) are given here by

$$m(0) = E(z(0)) = \bar{z}_0 \tag{41}$$

$$V(0) = E(z(0)z^T(0)) = \bar{z}_0\bar{z}_0^T + \Sigma_0 \tag{42}$$

By using Eqs. (5) and (30), it follows that

$$E(u(k)) = E(-[0 \quad F]z(k)) = -F_r m(k) \tag{43}$$

$$E(u(k)u^T(k)) - E(u(k))E(u^T(k))$$
$$= F_r V(k) F_r^T - F_r m(k) m^T(k) F_r^T \tag{44}$$

$$E(e(k)) = C_{cd}m(k) - y_d(k) \tag{45}$$

$$E(e(k)e^T(k)) = C_{cd}V(k)C_{cd}^T - C_{cd}m(k)y_d^T(k)$$
$$-y_d(k)m^T(k)C_{cd}^T + y_d(k)y_d^T(k) \tag{46}$$

where $k = 0, \dots, N$ and 0 denotes the zero matrix of appropriate dimensions in Eq. (43).

VI. Control System Design for a Linearized Aircraft Model

The application of the proposed method will be demonstrated in the design of a control system for the linearized longitudinal airframe of a V/STOL aircraft at 120 kn in steady level flight. The state-space matrices A_p, B_p, C_m, and C_p of the deterministic longitudinal airframe model were obtained from Ref. 13. The state-space model, including stochastic state disturbances and measurement noise, is represented by

$$dx_p(t) = [A_p x_p(t) + B_p u(t)]\,dt + G_p\,d\Phi(t) \tag{47}$$

$$dy_m(t) = C_m x_p(t)\,dt + H_p\,dW(t) \tag{48}$$

$$y(t) = C_p x_p(t) \tag{49}$$

where the matrices A_p, B_p, C_p, C_m, G_p, and H_p are given in the Appendix. The state vector is $x_p(t) = [\theta(t), q(t), ub(t), wb(t), de(t), dth(t), dn(t), fnp(t), hnp(t), qef(t)]^T \in R^{10}$ (pitch attitude, degree; pitch rate, degree/second; longitudinal body velocity, feet/second; normal body velocity, feet/second; tailplane angle, degree; throttle position, percent/100; nozzle angle, degree; engine fan speed, percent/100; engine compressor speed, percent/100; engine fuel flow, percent/100). The control input is $u(t) = [dec(t), dthc(t), dnc(t)]^T \in R^3$ (tailplane angle demand, degree; throttle demand, percent/100; nozzle angle demand, degree). The measured plant outputs are $C_m x_p(t)\,dt + H_p\,dW(t) = [\theta_m(t), v_m(t), \gamma_m(t), q_m(t)]^T\,dt \in R^4$ (measured pitch attitude, degree; measured airspeed, knots; measured flight path angle, degree; measured pitch rate, degree/second). The outputs to be controlled are $y(t) = [\theta(t), v(t), \gamma(t)]^T \in R^3$ (pitch attitude, degree; airspeed, knots; flight-path angle, degree).

For a reference command input $r(t) = [\theta_c(t), v_c(t), \gamma_c(t)]^T$, $t \in \Gamma$, the objective is to independently control the pitch angle, the airspeed, and the flight-path angle. To achieve this objective using integral augmentation, a subvector of the measured plant outputs, $[\theta_m(t), v_m(t), \gamma_m(t)]^T\,dt \in R^3$, given by

$$dy_o(t) = C_o\,dy_m(t) = C_o[C_m x_p(t)\,dt + H_p\,dW(t)]$$
$$= C_p x_p(t)\,dt + C_o H_p\,dW(t) \tag{50}$$

(C_o is given in the Appendix) is augmented with a dynamical system given by

$$dx_a(t) = [A_a x_a(t) - B_a r(t)]\,dt + B_a\,dy_o(t)$$
$$= [A_a x_a(t) - B_a r(t) + B_a C_p x_p(t)]\,dt + B_a C_o H_p\,dW(t) \tag{51}$$

where $A_a = 0_{3\times 3}$, $B_a = I_{3\times 3}$, and $Z_{a\times b}$ indicates an $a \times b$ matrix Z.

The state of the augmented system $x(t)$ consists of the state of the plant $x_p(t)$ and the state of the augmented dynamics $x_a(t)$,

$$x(t) = \left[x_p^T(t),\ x_a^T(t)\right]^T \in R^{13} \tag{52}$$

The measured output of the augmented system is

$$dy_\mu(t) = \left[\theta_m(t),\ v_m(t),\ \gamma_m(t),\ q_m(t),\ x_a^T(t)\right]^T dt \in R^7 \tag{53}$$

The augmented system is then described by

$$dx(t) = \left\{\begin{bmatrix} A_p & 0 \\ B_a C_p & A_a \end{bmatrix} x(t) + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -B_a \end{bmatrix} r(t)\right\} dt + \begin{bmatrix} G_p & 0 \\ 0 & B_a C_o H_p \end{bmatrix}\begin{bmatrix} d\Phi(t) \\ dW(t) \end{bmatrix}$$
$$= [Ax(t) + Bu(t) + B_r r(t)]\,dt + G\,dP(t) \tag{54}$$

$$dy_\mu(t) = \begin{bmatrix} C_m & 0 \\ 0 & I_{3\times 3} \end{bmatrix} x(t)\,dt + \begin{bmatrix} H_p \\ 0 \end{bmatrix} dW(t)$$
$$= C_\mu x(t)\,dt + H\,dW(t) \tag{55}$$

$$y(t) = [C_p \quad 0]x(t) = Cx(t) \tag{56}$$

where 0 denotes the zero matrix of appropriate dimensions in Eqs. (54–56). The augmented system is then connected with the corresponding LQG controller [Eqs. (4) and (5)] to form a closed-loop system described by Eqs. (17) and (18), where in this case

$$M = \begin{bmatrix} G_p & 0 \\ 0 & B_a C_o H_p \\ 0 & \Pi H \end{bmatrix}, \qquad B_c = \begin{bmatrix} B_r \\ 0 \end{bmatrix} \tag{57}$$

and where 0 denotes the zero matrix of appropriate dimensions in Eq. (57).

The data for the set Y in Sec. IV are as follows: $c_j = 1$, $j = 1, 2, 3$, and $\psi_d^{(j)}(t)$, $j = 1, 2, 3$, $t \in \Gamma$ (shown in Figs. 1–3) are the sampled step responses of second-order systems with (damping ratio, natural frequency) of (0.7, 2.5), (1.0, 1.5), and (1.0, 1.3), respectively. Since

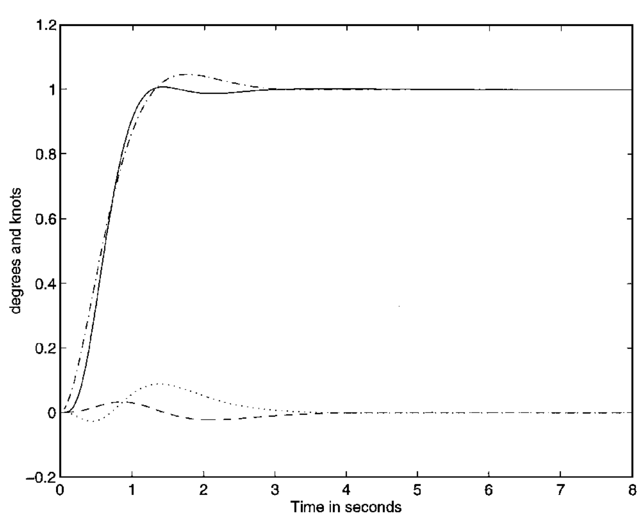

Fig. 1 Plots of $E(\theta(t))$ (——), $E(v(t))$ (– – –), $E(\gamma(t))$ (· · · ·), and $\psi_d^{(1)}(t)$ (– · –) for the case $r(t) = r^{(1)}(t)$, $t \in \Gamma$.

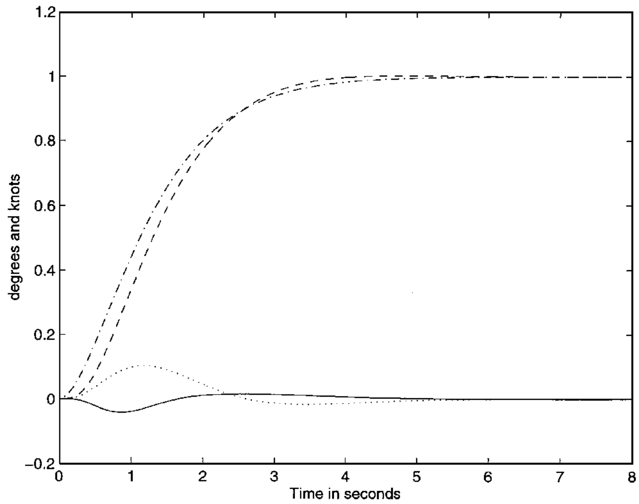

Fig. 2 Plots of $E(\theta(t))$ (——), $E(v(t))$ (– – –), $E(\gamma(t))$ (· · · ·), and $\psi_d^{(2)}(t)$ (– · –) for the case $r(t) = r^{(2)}(t), t \in \Gamma$.

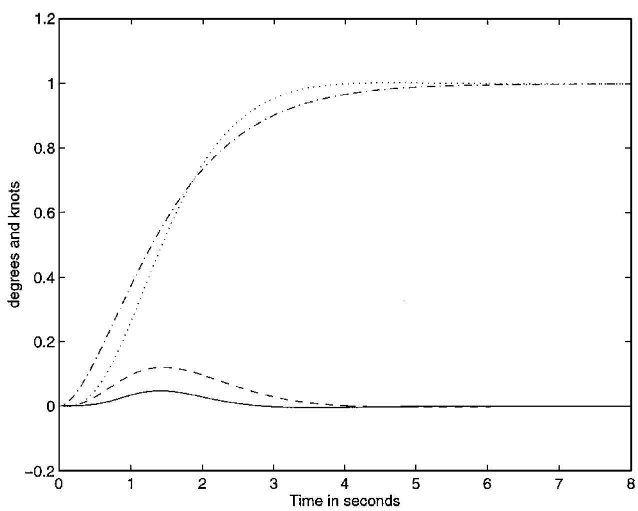

Fig. 3 Plots of $E(\theta(t))$ (——), $E(v(t))$ (– – –), $E(\gamma(t))$ (· · · ·), and $\psi_d^{(3)}(t)$ (– · –) for the case $r(t) = r^{(3)}(t), t \in \Gamma$.

$\psi_d^{(j)}(t)$, $j = 1, 2, 3$, $t \in \Gamma$ have different rise times, the pitch angle, airspeed, and flight-path angle will in effect be controlled approximately independently with different bandwidths. This is an important requirement for practical multivariable systems.[2]

VII. Computational Results

The functions $J_1(p,\ (r(\cdot),\ y_d(\cdot)))$ and $J_2(p)$ [Eqs. (26) and (27)] are used with the following data: $n = 3$, $m = 13$, $q = 3$, $v = 7$, $emax_i = 0.12$, $i = 1, 2, 3$, $erms_1 = 0.11$, $erms_2 = 0.10$, $erms_3 = 0.13$, $uvar_1 = 1.3$, $uvar_2 = 0.9$, $uvar_3 = 1.3$, $maxr = 31$ rad/s, $maxf = 50$ rad/s, $s_{\min} = 0.575$, $a_i = 10^4$, $i = 1, 2, 3$, $a_4 = 10^2$, $a_5 = 10^2$, and $a_6 = 10$.

In addition, the following numerical data are used: $T = 0.02$ s, time horizon $= NT = 8$ s, that is, $N = 400$, $\zeta_0 = -2$, $\Delta\zeta = 5/99$, $N_\omega = 99$, that is the frequency varies from 0.01 rad/s to 10^3 rad/s, $m(0) = \bar{z}_0 = 0_{26\times 1}$, $V(0) = \Sigma_0 + \bar{z}_0\bar{z}_0^T = \text{diag}(0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0_{1\times 16})$ and the initial values of the design parameter matrices were chosen as $Q_0 = \text{diag}\ (0_{1\times 10}, 1, 1, 1)$ and $L_0 = \text{diag}(9, 0, 1, 4, 0, 36, 0, 0, 0, 4, 1, 0, 0)$.

The Nelder–Mead algorithm as implemented by the function FMINS of the PC-MATLAB® Optimization Toolbox[9] was then applied until $J(p)$ [Eq. (28)] reached the value of zero in double precision. Thus, all the specifications have been exactly met. The computed design parameter matrices were as follows: $Q = \text{diag}(0.23455, 1.0671, 30.969, 0.34451, 1.4511, 14.413, 0.79643, 12.628, 24.353, 31.901, 437.6, 177.37, 329.17)$ and $L = \text{diag}(916.59, 1.6269, 80.623, 1099.0, 1.4545, 1828.2, 2.9783, 15.807, 3.3075, 4894.8, 194.15, 9.8119, 1.1484)$. The resulting gain matrices F and Π can be obtained from Eqs. (6–9) and will not be given here due to space constraints.

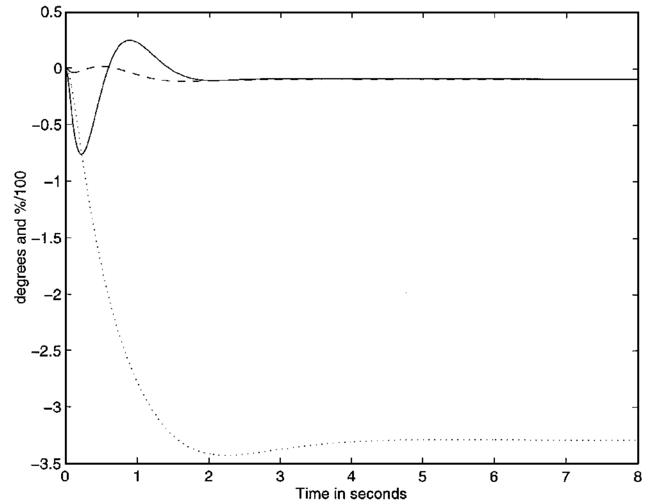

Fig. 4 Plots of the expected control inputs, $E(u_1(t))$ (——), $E(u_2(t))$ (– – –), and $E(u_3(t))$ (· · · ·) for the case $r(t) = r^{(1)}(t), t \in \Gamma$.

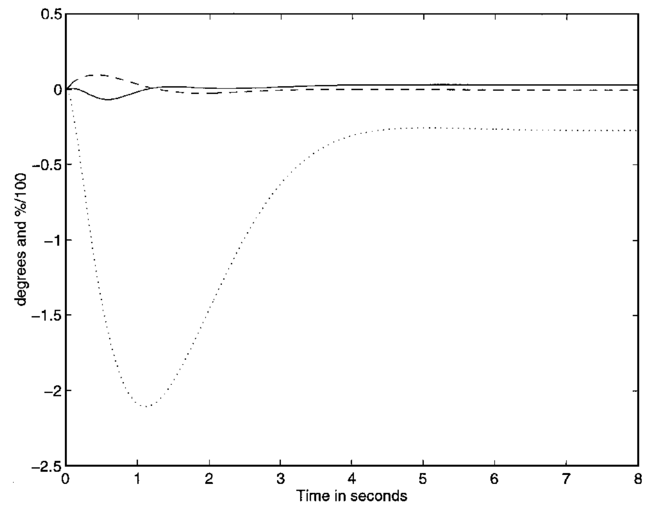

Fig. 5 Plots of the expected control inputs, $E(u_1(t))$ (——), $E(u_2(t))$ (– – –), and $E(u_3(t))$ (· · · ·) for the case $r(t) = r^{(2)}(t), t \in \Gamma$.

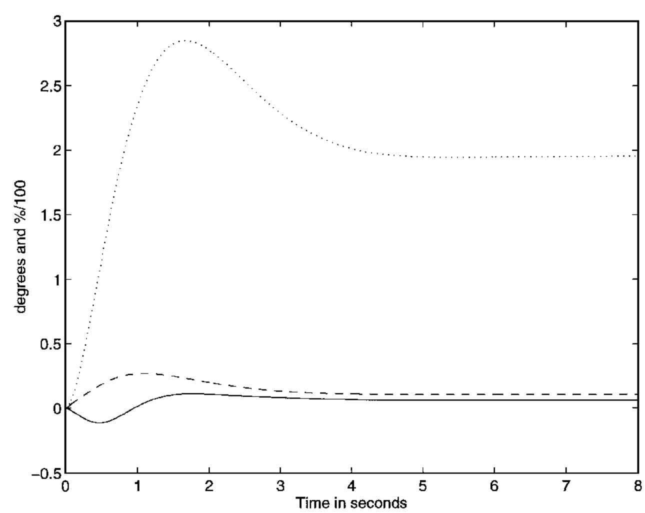

Fig. 6 Plots of the expected control inputs, $E(u_1(t))$ (——), $E(u_2(t))$ (– – –), and $E(u_3(t))$ (· · · ·) for the case $r(t) = r^{(3)}(t), t \in \Gamma$.

For all $(r(\cdot),\ y_d(\cdot)) \in Y$ the expected value of the controlled variables are shown in Figs. 1–3, respectively, whereas the expected value of the corresponding control inputs are shown in Figs. 4–6, respectively. In addition, a plot of the root mean square tracking errors for the case $r(k) = r^{(1)}(k)$, $k = 0, \ldots, N$, is shown in Fig. 7.

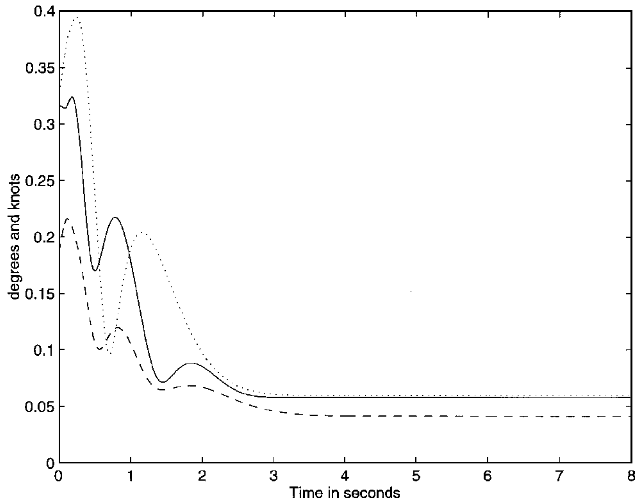

Fig. 7 Plots of $(E(e_1^2(t)))^{1/2}$ (——), $(E(e_2^2(t)))^{1/2}$ (- - -), and $(E(e_3^2(t)))^{1/2}$ (· · · ·) for the case $r(t) = r^{(1)}(t)$, $t \in \Gamma$.

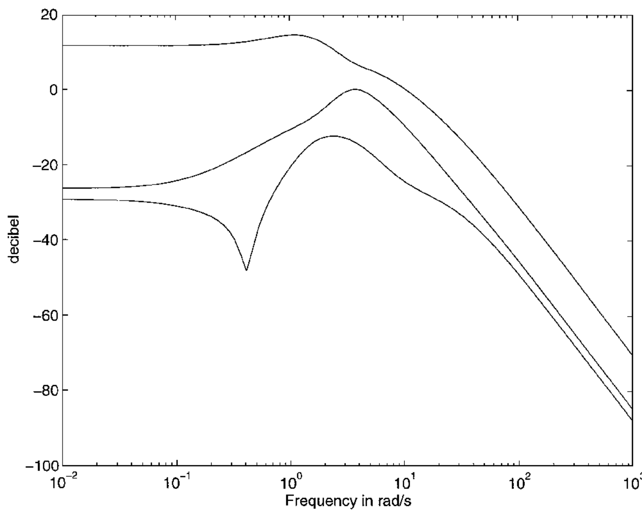

Fig. 8 Plots of the singular values of $H_{ru}(j\omega)$ vs frequency ω.

For the cases $r(k) = r^{(j)}(k)$, $j = 2, 3$, $k = 0, \ldots, N$, the plots are similar to that in Fig. 7.

The eigenvalues of $(A - BF)$ are as follows: -30.924, $-28.998 \pm j7.7498$, $-5.0536 \pm j3.2474$, $-2.4964 \pm j3.0148$, -6.6106, $-1.2098 \pm j\ 0.94425$, -5.2479, -4.2102, and -1.6952.

The eigenvalues of $(A - \Pi C_\mu)$ are as follows: -43.294, $-20.796 \pm j21.667$, -19.999, $-6.536 \pm j4.9004$, -13.944, -13.987, -7.3796, -5.8658, -3.305, -1.4696, and -4.9993.

The poles of the compensator $\Lambda(s)$ are as follows: $-29.591 \pm j32.361$, -49.865, $-33.877 \pm j9.3863$, -13.918, $-10.923 \pm j2.6987$, $-7.5953 \pm j4.1593$, -4.6769, -1.4868, and -3.3768.

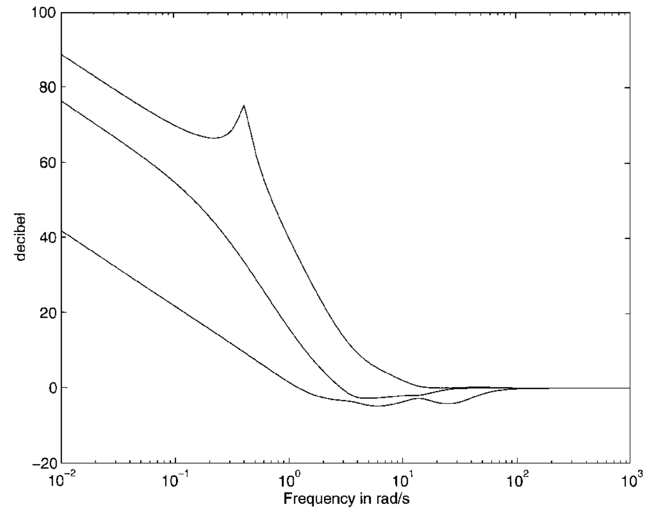

Fig. 9 Plots of the singular values of $[I + T_{in}(j\omega)]$ vs frequency ω.

The singular values of the transfer function $H_{ru}(j\omega)$ from the reference command to the control input, $U(s) = H_{ru}(s)R(s)$, are shown in Fig. 8, whereas the singular values of the return difference transfer function matrix at the plant input are shown in Fig. 9. Defining d_0 as[12]

$$d_0 = \min_{\omega \in \Omega} \left(\min_{k \in \{1, \ldots, q\}} (\sigma_k(I + T_{in}(j\omega))) \right) \tag{58}$$

we find that for this case $d_0 = 0.57571 > s_{\min}$, implying that constraint (25) has been satisfied.

VIII. Conclusions

This work has proposed a dynamic controller design method for stochastic systems such that time and frequency domain specifications are met.

An important feature of the proposed method is that, whenever possible, only feasible controllers are computed for which all the system specifications and constraints are met. The method has been demonstrated via an example dealing with the design of a flight control system. The results obtained, part of which are presented in Figs. 1–9, demonstrate the effectiveness of the proposed method.

The question of existence of solutions to $J(p) = 0$, $p \in R^{2m}$, is out of the scope of this work. The mapping from p to $J(p)$ is too complicated for guaranteeing the existence of p^0. Nevertheless, the method of computing a feasible controller, which is used here, has been successfully applied to a variety of different problems; see, for example, Refs. 14–20.

Appendix: Plant Data

The plant matrices are as follows:

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.393E-5 & -0.6917 & 0.1692 & -0.1811E-1 & -7.340 & 0 & 0.158 & 62.92 & -66.36 & -11.97 \\ -0.5561 & -0.4872 & -0.4164E-1 & -0.8246E-1 & -0.4258E-1 & 0 & -0.3736 & 14.58 & 5.664 & 2.947 \\ -0.7811E-1 & 3.469 & 0.1801E-1 & -0.2949 & -0.4084 & 0 & -0.1201 & -28.13 & -7.183 & -3.899 \\ 0 & 0 & 0 & 0 & -20 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4.999 & 0 & 0 & 0 \\ 0 & 0 & -0.1131E-4 & -0.2486E-5 & 0 & 0 & 0 & -3.747 & 2.536 & 1.124 \\ 0 & 0 & -0.1179E-4 & -0.1045E-5 & 0 & 0 & 0 & -0.6261E-3 & -2.711 & 0.8217 \\ 0 & 0 & 0.8166E-3 & 0.1073E-3 & 0 & 15.58 & 0 & -58.2 & 0 & -13.33 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5863 & 0.824E-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.3934E-1 & -0.2799 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5863 & 0.824E-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.3934E-1 & -0.2799 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $C_o = [I_{3\times 3} \;\; 0_{3\times 1}]$, $G_p = 0.05 I_{10\times 10}$, and $H_p = 0.02 I_{4\times 4}$.

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